Tolerance Limits and Confidence Limits on Reliability for the Two-Parameter Exponential Distribution

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Tolerance limits and confidence limits on reliability, which closely approximate exact limits, are proposed for the two-parameter exponential distribution. These approximations have the advantage that solutions to both the tolerance limit problem and the confidence limit problem can be written explicitly.

KEY WORDS
Tolerance Limits
Confidence Limits
Reliability
Exponential Distribution

1. INTRODUCTION

In a recent paper Guenther, Patil and Uppuluri [3] tabulated one-sided $p$-content tolerance factors for the two parameter exponential distribution. Other references to this problem and the related problem of confidence limits on reliability are provided in [1], [2], and [4]. In the present paper, close approximations are provided which apply to both the tolerance limit and confidence limit problems.

Let $X_1 < X_2 < \cdots < X_r$ denote the $r$ smallest order statistics in a random sample of size $n$ from a two-parameter exponential distribution defined by the density function

$$f(x; \mu, \sigma) = \frac{1}{\sigma} \exp \left[ -\frac{x - \mu}{\sigma} \right],$$

$x > \mu, \sigma > 0.$ \hfill (1)

Further let $U_i = U_i(X_1, X_2, \cdots, X_r)$ denote a statistic. Then an observed value, $u_i$, is called a lower one-sided $p$-content tolerance limit at level $\gamma$ if

$$P \left[ \int_{U_i} f(x; \mu, \sigma) \, dx \geq \beta \right] = \gamma \hfill (2)$$

It is possible to base tolerance limits on the complete sufficient statistics $T = X_1$ and

$$S = \sum_{i=1}^{r} X_i + (n - r)X_r - nX_1.$$  

In particular, we take

$$U_i = T + k_1S$$ \hfill (3)

where $k_1$ is a solution (independent of $\mu$ and $\sigma$) to

$$P[\frac{(T - \mu)}{\sigma} + k_1S/\sigma \leq \ln \beta] = \gamma. \hfill (4)$$

This procedure can also be used to find confidence limits on reliability $R(T) = P(X > T)$. In particular, for observed values $t$ and $s$, and for a specified confidence level $\gamma$, determine the value of $\beta = \beta_\gamma$ such that $u_i = t + k_1s = \tau$. Then define $R_\gamma(\tau) = \beta_\gamma$. It was noted by Guenther et al [3] that such a limit can exceed 1. One possible solution would be to take min $(1, \beta_\gamma)$ in which case the limit is conservative with confidence level at least $\gamma$. The approximations given here apply to either complete or censored samples, and have the advantage of providing convenient closed-form solutions to both the tolerance limit and confidence limit problems.

It will be shown in section 2 that for $n \leq \ln (1 - \gamma)/\ln \beta$ an exact solution to equation (4) is given by

$$k_1 = \frac{1}{n} \left[ 1 - \frac{\beta^{1/n} - (1 - \gamma)^{1/n}}{(1 - \gamma)^{1/n}} \right].$$ \hfill (5)

Although for $n > \ln (1 - \gamma)/\ln \beta$ an exact solution cannot be given explicitly, equation (5) provides a very good approximation for larger $n$ (see Table 1). The corresponding lower $\gamma$ level confidence limit for reliability is

$$R_\gamma(\tau) = (1 - \gamma)^{1/n} \left[ 1 - n(\tau - 1)/S \right]^{-1/n}. \hfill (6)$$

The following formulas, which are also derived in
section 2, give very good approximations for larger sample sizes:

\[
k_i = \frac{1}{n} \left[ m(\beta) \right]
\]

where \( m(\beta) = \left( 1 + n \ln \beta \right) / (r - 5/2) \), and

\[
R_i(r) = \exp \left\{ -\frac{1}{r} - \frac{(r/n) \left[ \left( r - 5/2 \right) / \alpha \right]}{a} \right\}
\]

where \( Y = \frac{Z}{\sqrt{\alpha}} \), \( \alpha = \left( 1 - \frac{Z^2}{r} \right) \), and \( Z \) is the \( \gamma \)-quantile of the standard normal distribution.

Although equations (7) and (8) work fairly well for moderately small sample sizes, the recommended procedure would be to use equations (5) and (6) for small sample sizes and equations (7) and (8) for large sample sizes. Guidelines are provided in Table 1 which indicate the maximum sample size \( n \), such that (5) is exact when \( n \leq n_0 \). Furthermore, Table 1 provides the maximum value \( r \), such that (5), when used as an approximation, provides a probability level which is correct to two decimal places when \( r \leq r_1 \).

It is also possible to obtain approximate upper one-sided \( \beta \)-content tolerance factors, \( k_2 \), at level \( \gamma \) by replacing \( \gamma \) with \( 1 - \gamma \) and \( \beta \) with \( 1 - \beta \) in (7). Similarly, it is possible to obtain approximate upper one-sided \( \gamma \) level confidence limits, \( R_2(r) \), for reliability by replacing \( \gamma \) with \( 1 - \gamma \) in (8).

The methods will be illustrated by an example of Grubbs [2] involving data for nineteen military personnel carriers which failed in service. The observed values for a complete sample are \( t = 162 \) and \( s = 15869 \). Suppose a lower \( \beta = .8 \) tolerance limit at level \( \gamma = .95 \) is desired. From Table 1 we find that \( n = 19 > n_0 = 13 \), but also \( r = 19 < 74 = r_1 \). Hence equation (5) is not exact, but provides a good approximation in this case. The factor is \( k_1 = \sqrt{\left( 1/19 \right) \left( 1 - (0.8)^{19/19} \right) = 0.0035 \) and the limit is \( u_2 = 162 + (0.0035)(15869) = 218 \) miles. Using (6) we obtain a lower \( \beta = .95 \) confidence limit \( R_2(200) = 192.06 \) for \( R(200) \). It is interesting to note that both limits are identical to the corresponding limits obtained by Grubbs et al [3] using their tabulated factors. An approximate upper \( \beta \) tolerance factor for \( \gamma = .9 \) is obtained by replacing both \( \beta \) and \( \gamma \) with \( .1 \) in (7). Specifically,

\[
m(\beta) = \left( 1 + 19 \ln .1 \right) / (19 - 5/2) = -2.59
d\]

\[
k_2 = \left( 1/19 \right) \left( -(-2.59) \right) = 1.765.
\]

The corresponding tolerance limit is \( u_2 = 162 + (.1765)(15869) = 2963 \) miles. For comparison, Guenther et al [3] obtained \( k_2 = .1757 \) and \( u_2 = 2950 \) miles. The relative error is less than 4%.

2. DERIVATIONS

Let \( U = 2n(T - \mu)/\sigma \) and \( V = 2 S / \sigma \). It is well known that \( U \) and \( V \) are independently distributed, chi-square with 2 and 2\( (r - 1) \) degrees of freedom, respectively. If we define

\[
G(\lambda; r, n, \beta) = P[U/2n + \lambda V/2 \leq -\ln \beta],
\]

then \( k_1 \), as defined by (4) would be a solution \( \lambda = k_1 = k_1(r, n, \beta, \gamma) \) to the equation \( \gamma = G(\lambda; r, n, \beta) \). It is easily shown that, for \( r, n \) and \( \beta \) fixed, (9) is a monotonic decreasing function of \( \lambda \), and that \( G(0; r, n, \beta) = 1 - \beta^r \). It follows that \( k_1 \leq 0 \) if and only if \( n \leq \ln (1 - \gamma) / \ln \beta \). In this case the equation becomes \( \gamma = 1 - \beta^r (1 - n k_1)^{-1} \) which yields the explicit solution (5). Exact solutions \( k_1 \) for \( n > \ln (1 - \gamma) / \ln \beta \) could also be obtained from (9), although the derivations are tedious and the numerical solutions would involve iteration or trial and error. Approximations which can be written explicitly might be preferable. Guidelines for using (5) as an approximation were obtained by inserting values of \( \lambda = k_1 \), for different values of \( r \), into (9) and tabulating the maximum value, \( n_0 \), such that the true probability level is correct to two decimal places when \( n \leq n_0 \). Equations (7) and (8) are based upon a normal approximation of the distribution of \( W = 2(r/n)(U/2 + n \ln \beta) \). Since \( \gamma = P[W \leq -rk_1] \) it suffices to consider this variable. It can be shown that \( W \) has an asymptotic normal distribution with mean \( \ln \beta \) and variance \( (1/n)(\ln \beta)^2 / \rho \) where \( r/n \to \rho \) as \( n \to \infty \). It is possible to derive the exact mean and variance of \( W \):

\[
E(W) = \mu(\beta) = (r/n)(1 + n \ln \beta) / (r - 2)
\]

and

\[
Var(W) = \sigma^2(\beta) = (r/n)^2 \mu^2(\beta) / (r - 3).
\]

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The small-sample approximation is improved, although the asymptotic distribution remains the same, if we use the exact moments (10) and (11). Additional improvement results if we replace \( r - 2 \) with \( r - 5/2 \) in (10) and \( r - 3 \) with \( r \) in (11). The resulting approximate factor is given by (7). If we equate (3) to \( \tau \), using (7), then solve explicitly for \( R_\alpha(\tau) = \beta \) we obtain (8).

Comparisons are made in Table 2 with exact factors, the approximate factors proposed in section 1, and approximate factors based upon a ratio of chi-square quantiles as given by Guenther et al [3].

Although Guenther et al [3, p. 334] point out that their results also apply to the case of censored data (with \( r \) replacing \( n \)), great care must be exercised in this case. Their tabulations apply directly to the complete sample case \( (r = n) \), and are of the form \( k_\alpha = k_{\alpha}(n, n, \beta, \gamma) \) or equivalently \( k_\alpha(r, r, \beta, \gamma) \). In the case of a censored sample \( (r < n) \), it can be shown that the appropriate factor is of the form \( k_\alpha(r, n, \beta, \gamma) = (r/n)k_\alpha(r, r, \beta^n, \gamma) \). This follows from (9) with \( \gamma = G(k_\alpha; r, n, \beta) = G(k_\alpha^*; r, r, \beta^*) \) where \( k_\alpha^* = (n/r)k_\alpha \) and \( \beta^* = \beta^n \). Thus it is necessary to replace \( n \) with \( r \) and \( \beta \) with \( \beta^n \) in their table and then multiply the resulting factor by \( r/n \). The same remarks hold for their table of upper tolerance factors, except that \( \beta \) should be replaced by \( 1 - (1 - \beta)^{1/n} \). These remarks also explain why we have used \( \beta^n \) rather than \( \beta \) as a heading for values of \( r \) in Table 1.

3. ACKNOWLEDGEMENT

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REFERENCES
